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Pure gauge configurations and solutions to fermionic superstring field theory equations of motion

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Abstract

Recent results on solutions to the equation of motion of the cubic fermionic string field theory and an equivalence of nonpolynomial and cubic string field theory are discussed. To have the possibility of dealing with both GSO(+) and GSO(−) sectors in the uniform way, a matrix formulation for the NS fermionic SFT is used. In constructions of analytical solutions to open-string field theories truncated pure gauge configurations parametrized by wedge states play an essential role. The matrix form of this parametrization for NS fermionic SFT is presented. Using the cubic open superstring field theory as an example we demonstrate explicitly that for the large parameter of the perturbation expansion these truncated pure gauge configurations give divergent contributions to the equations of motion on the subspace of the wedge states. The perturbation expansion is corrected by adding extra terms that are just those necessary for the equation of motion contracted with the solution itself to be satisfied.

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1. Introduction

It is well known that string field theories (SFT) describe an infinite number of local fields. Just for this reason finding nontrivial solutions to classical SFT is a rather nontrivial problem. This is a reason why the Schnabl construction of the tachyon solution in the Witten open bosonic SFT [1] attracts a lot of attention [2]. It turns out that the tachyon solution is closely related to pure gauge solutions. More precisely, Schnabl's solution is the regularization of a singular limit of a pure gauge configuration [2, 7]. The presence of pure gauge solutions in the bosonic SFT is related to the Chern–Simons form of the Witten cubic action. The Schnabl solution is distinguished by the fact that it describes a true vacuum of SFT, i.e. a vacuum on which the Sen conjecture is realized. Since the pure gauge solutions do not shift the vacuum energy, the correct shift of the vacuum energy by the Schnabl solution is rather a nontrivial fact, and its deep origin is still unclear to us.

[3] Deceased.

The purpose of this paper is to present recent results concerning the generalization of the Schnabl solution to the fermionic case.

It is natural to expect that a solution being a singular limit of a pure gauge solution also exists in the cubic super SFT (SSFT) [3, 5]. But for the superstring case, there is no *a priori* reason to deal with the Sen conjecture, since the perturbative vacuum is stable (there is no tachyon). However, a nontrivial (not pure gauge) solution to the SSFT equation of motion (EOM) does exist [15]. The physical meaning of this solution is still unclear. It may happen that it is related to a spontaneous supersymmetry breaking (compare with [4]).

There is also a nonpolynomial formulation of the SSFT [12]. A solution of the equation of motion in the nonpolynomial SSFT has been obtained in [16, 17]. This solution became clear after the realization of an explicit relation between solutions to the cubic and nonpolynomial SSFTs [14]. These theories include only the GSO(+) sector of the NS string. There are also two versions of the NS fermionic SFT that includes both GSO(+) and GSO(−) sectors, cubic [8] and nonpolynomial [12]. Just the NS fermionic SFT with two sectors is used to describe non-BPS branes. The Sen conjecture has been checked by the level truncation for the nonpolynomial and cubic cases in [11] and [8], respectively. A solution to the equation of motion of the cubic SFT describing the NS string with both GSO(+) and GSO(−) sectors has been constructed in [9]. On this solution the Sen conjectures take place.

To make the construction of the solution [9] more clear it is useful to incorporate a matrix version of NS fermionic SFT with GSO(+) and GSO(−) [10]. In the matrix formulation, an explicit relation between solutions to the cubic and nonpolynomial theories becomes more clear, and it gives an explicit formula for solutions to BSZ theory [11] via solutions [9] to ABKM theory [8].

The Schnabl solution Ψ consists of two pieces and is defined by the limit,

$$\Psi = \lim_{N\to\infty}\left[\sum_{n=0}^{N}\psi_n' - \psi_N\right], \tag{1}$$

where the states ψ_n' defined for any real $n \geqslant 0$ are made of the wedge state [18–20].

Generally, in our calculations, in particular in numerical calculations, we are able to check the equation of motion in a weak sense contracted with a vector C:

$$\langle C, Q\Psi + \Psi \star \Psi\rangle = 0. \tag{2}$$

A weak validity of the equation of motion depends on the domain of contracting vectors C. It was shown [2] that the string field Ψ in (1) solves the equation of motion of Witten's SFT contracted with any state C in the Fock space with a finite number of string excitations. In the course of this calculation ψ_N piece does not contribute.

On the other hand to check the Sen conjecture, one has to use the equation of motion contracted with a solution itself. The ψ_N piece in (1) is necessary for the equation of motion contracted with the solution itself to be satisfied [7, 14].

The first piece in the Schnabl constructions (1) is related to a perturbative expansion over a parameter λ of a pure gauge configuration:

$$\Psi_N(\lambda) = \sum_{n=0}^{N}\lambda^{n+1}\psi_n'. \tag{3}$$

It is worth stressing that $\Phi_\infty(1)$ fails to be a solution of the equation of motion when contracting with a wide class of states that are the building blocks of the solution, namely with the wedge states ψ_n:

$$\langle\psi_m, Q\Phi_\infty(1) + \Phi_\infty(1)\star\Phi_\infty(1)\rangle \neq 0, \tag{4}$$

and this is an origin why the pure gauge configurations $\Phi_\infty(1)$ do not solve the equation of motion contracted with these configurations themselves. We demonstrate this fact explicitly for the cubic open SSFT.

It is possible to correct the perturbation expansion $\Phi_\infty(1)$ by adding extra terms ψ_N. These are just the terms that have been used previously to provide that the equation of motion contracted with the solution itself be satisfied [15].

The paper is organized as follows.

In section 2 a matrix formulation for the NS fermionic SFT is presented.

In section 3 perturbative parameterizations of special pure gauge configurations are presented. These pure gauge configurations are used in the Erler SSFT solution [15] and in the tachyon fermion solution [9].

In section 4 we demonstrate that the $\lambda = 1$ limit of these pure gauge solutions is, in fact, a singular point and we use a simple prescription to correct divergences. We show that this prescription gives the same answer as the requirement of the validity of the equations of motion contracted with the solution itself.

In section 5, we contribute to a discussion [14] of the classical equivalence of the nonpolynomial theory of Berkovits, Sen and Zwiebach [11], and the cubic theory of Belov, Koshelev and two of us [8].

2. Cubic SFT for fermion string with GSO(+) and GSO(−) sectors in matrix notations

The action for covariant SSFT with GSO(+) and GSO(−) sectors was proposed in [8]:

$$S[\Phi_+,\Phi_-] = -\frac{1}{g_0^2}\left[\frac{1}{2}\langle Y_{-2}\Phi_+, Q\Phi_+\rangle + \frac{1}{3}\langle Y_{-2}\Phi_+,\Phi_+,\Phi_+\rangle \right.$$
$$\left. + \frac{1}{2}\langle Y_{-2}\Phi_-, Q\Phi_-\rangle - \langle Y_{-2}\Phi_+,\Phi_-,\Phi_-\rangle\right]. \tag{5}$$

The equations of motion read ($\star$ stands for Witten's string field product)

$$Q\Phi_+ + \Phi_+\star\Phi_+ - \Phi_-\star\Phi_- = 0, \tag{6}$$

$$Q\Phi_- + \Phi_+\star\Phi_- - \Phi_-\star\Phi_+ = 0. \tag{7}$$

The string fields Φ_+ and Φ_- have definite and opposite Grassman parity, to be fixed below. The parity $|\Phi|$ leads to the Leibniz rule,

$$Q(\Phi\star\Psi) = Q\Phi\star\Psi + (-)^{|\Phi|}\Phi\star Q\Psi. \tag{8}$$

It is useful to introduce matrix notations [10] by tensoric string fields and operators with appropriate 2×2 matrices. In this notations, the action (5) reads

$$S[\widehat{\Phi}] = -\frac{1}{g_0^2}\left[\frac{1}{2}\langle \widehat{Y}_{-2}\widehat{\Phi}, \widehat{Q}\widehat{\Phi}\rangle + \frac{1}{3}\langle \widehat{Y}_{-2}\widehat{\Phi},\widehat{\Phi},\widehat{\Phi}\rangle\right], \tag{9}$$

and the string field $\widehat{\Phi}$ is given by [10]

$$\widehat{\Phi} = \Phi_+\otimes\sigma_3 + \Phi_-\otimes\, \mathrm{i}\sigma_2, \tag{10}$$

and

$$\widehat{Q} = Q\otimes\sigma_3, \qquad \widehat{Y}_{-2} = Y_{-2}\otimes\sigma_3, \tag{11}$$

where σ_i are Pauli matrices.

The parity assignment and σ_i algebra lead to the Leibnitz rule:

$$\widehat{Q}(\widehat{\Phi}\star\widehat{\Psi}) = (\widehat{Q}\widehat{\Phi})\star\widehat{\Psi} + (-)^{|\widehat{\Phi}|}\widehat{\Phi}\star(\widehat{Q}\widehat{\Psi}), \tag{12}$$

where

$$|\widehat{\Phi}| \equiv |\Phi_+|. \tag{13}$$

The equations of motion (6) in the matrix notations read

$$\widehat{Q}\widehat{\Phi} + \widehat{\Phi}\star\widehat{\Phi} = 0. \tag{14}$$

If $\widehat{\Phi}$ is a nontrivial solution of (14), then it has to be Grassman odd, $|\widehat{\Phi}| = 1$. The pure gauge solution of (14) is

$$\widehat{\Phi} = \widehat{\Omega}^{-1}\star\widehat{Q}\widehat{\Omega} = -\widehat{Q}\widehat{\Omega}\star\widehat{\Omega}^{-1}, \tag{15}$$

for $\widehat{\Phi}$ to be odd $\widehat{\Omega}$ has to be even $|\widehat{\Omega}| = 0$, and it has an expansion

$$\widehat{\Omega} = \Omega_+\otimes I + \Omega_-\otimes\sigma_1. \tag{16}$$

3. Perturbative pure gauge solution

3.1. Perturbative expansion in matrix notations

In this section, we find a solution of the equation of motion (14). We find the solution as a series in some parameter λ, i.e., let us suppose $\widehat{\Phi}$ to be a series in some λ,

$$\widehat{\Phi}^{\lambda} = \sum_{n=0}^{\infty}\lambda^{n+1}\widehat{\phi}_n, \tag{17}$$

and put this expansion in equation of motion (14). In the first order in λ we have

$$\widehat{Q}\widehat{\phi}_0 = 0. \tag{18}$$

We choose a solution to (18) as

$$\widehat{\phi}_0 = \widehat{Q}\widehat{\phi}, \tag{19}$$

where

$$\widehat{\phi} = \phi_+\otimes I + \phi_-\otimes\sigma_1, \tag{20}$$

ϕ_+ and ϕ_- are the components of the gauge field $\widehat{\phi}$, and they belong to GSO(+) and GSO(−) sectors, respectively. The Grassman parities of ϕ_+ and ϕ_- are opposite.

In the second order in λ, we have

$$\widehat{Q}\widehat{\phi}_1 + \widehat{\phi}_0\star\widehat{\phi}_0 = 0. \tag{21}$$

For $\widehat{\phi}_0$ in the form (19) we get (also we used the Leibnitz rule (12) for $\widehat{Q}$)

$$\widehat{Q}\widehat{\phi}_1 + \widehat{Q}\widehat{\phi}\star\widehat{Q}\widehat{\phi} = \widehat{Q}(\widehat{\phi}_1 - \widehat{Q}\widehat{\phi}\star\widehat{\phi}) = 0, \tag{22}$$

due to $|\widehat{\phi}| = 0$ we get minus. The solution of equation (21) is

$$\widehat{\phi}_1 = \widehat{Q}\widehat{\phi}\star\widehat{\phi}. \tag{23}$$

In this scheme, we get

$$\widehat{\phi}_n = \widehat{Q}\widehat{\phi}\star\widehat{\phi}^n, \tag{24}$$

then $\widehat{\Phi}$ is

$$\widehat{\Phi}^{\lambda} = \sum_{n=0}^{\infty}\lambda^{n+1}\widehat{Q}\widehat{\phi}\star\widehat{\phi}^n = \lambda\widehat{Q}\widehat{\phi}\frac{1}{1-\lambda\widehat{\phi}}. \tag{25}$$

The perturbative solution has the pure gauge form (15). Indeed, let us introduce $\widehat{\Omega} = 1 - \lambda\widehat{\phi}$, then (25) is

$$\widehat{\Phi}^{\lambda} = -\widehat{Q}(1 - \lambda\widehat{\phi}) \star (1 - \lambda\widehat{\phi})^{-1} = -\widehat{Q}\widehat{\Omega} \star \widehat{\Omega}^{-1}. \tag{26}$$

This expression can be written through $\phi_{\pm}$ as

$$\widehat{\Phi}^{\lambda} = (Q\phi_+ \otimes \sigma_3 + Q\phi_- \otimes \, \mathrm{i}\sigma_2)\frac{1}{(1-\lambda\phi_+)^2 - \lambda^2\phi_-^2}((1-\lambda\phi_+) \otimes I + \lambda\phi_- \otimes \sigma_1). \tag{27}$$

Picking out GSO(+) and GSO(−) sectors we get

$$\Phi_+^{\lambda} = \frac{\lambda}{2}Q(\phi_+ + \phi_-)\frac{1}{1-\lambda(\phi_+ + \phi_-)} + \frac{\lambda}{2}Q(\phi_+ - \phi_-)\frac{1}{1-\lambda(\phi_+ - \phi_-)}, \tag{28}$$

$$\Phi_-^{\lambda} = \frac{\lambda}{2}Q(\phi_+ + \phi_-)\frac{1}{1-\lambda(\phi_+ + \phi_-)} - \frac{\lambda}{2}Q(\phi_+ - \phi_-)\frac{1}{1-\lambda(\phi_+ - \phi_-)}. \tag{29}$$

This result agrees with [9].

3.2. Initial data and perturbative expansion in components

Here we choose ϕ_+ and ϕ_- in the following form [9]:

$$\phi_+ = B_1^L c_1|0\rangle, \tag{30}$$

$$\phi_- = B_1^L \gamma_{\frac{1}{2}}|0\rangle. \tag{31}$$

Then Φ_+^{λ} and Φ_-^{λ} will have the form

$$\Phi_+^{\lambda} = \sum_{n=0}^{\infty}\lambda^{n+1}\phi'_n, \tag{32}$$

$$\phi'_0 = \big(-K_1^R c_1 - B_1^R\big(c_0c_1 + \gamma_{1/2}^2\big)\big)|0\rangle, \tag{33}$$

$$\phi'_n = c_1|0\rangle \star |n\rangle \star K_1^L B_1^L c_1|0\rangle + \gamma_{\frac{1}{2}}|0\rangle \star |n\rangle \star K_1^L B_1^L \gamma_{\frac{1}{2}}|0\rangle, \qquad n > 0, \tag{34}$$

$$\Phi_-^{\lambda} = \sum_{n=0}^{\infty}\lambda^{n+1}\psi'_n, \tag{35}$$

$$\psi'_0 = \left(-K_1^R\gamma_{\frac{1}{2}} + B_1^R\left(c_1\gamma_{-\frac{1}{2}} - \frac{1}{2}c_0\gamma_{\frac{1}{2}}\right)\right)|0\rangle, \tag{36}$$

$$\psi'_n = \gamma_{\frac{1}{2}}|0\rangle \star |n\rangle \star K_1^L B_1^L c_1|0\rangle + c_1|0\rangle \star |n\rangle \star K_1^L B_1^L \gamma_{\frac{1}{2}}|0\rangle, \qquad n > 0. \tag{37}$$

4. The $\lambda = 1$ limit

In this section, we examine the $\lambda = 1$ limit of the pure gauge solutions (27). It is known that this is a singular point for the pure gauge solution [2, 7, 16].

We consider for the transparency a pure GSO(+) sector and the equation of motion for the string field Φ_+ is

$$Q\Phi_+ + \Phi_+ \star \Phi_+ = 0. \tag{38}$$

We start with the pure gauge solution of (38) given by formulae (32)–(37) with $\lambda < 1$ and initial date $\phi_- = 0$. The explicit form of this solution is

$$\Phi_+(\lambda) = \sum_{n=0}^{\infty} \lambda^{n+1}\varphi'_n + \lambda\Gamma, \qquad |\lambda| < 1, \tag{39}$$

where

$$\begin{aligned} \Gamma &= B_1^L \gamma_{1/2}^2|0\rangle, \\ \varphi'_0 &= -\left(K_1^R c_1 + B_1^R c_0 c_1\right)|0\rangle, \\ \varphi'_n &= c_1|0\rangle \star |n\rangle \star K_1^L B_1^L c_1|0\rangle n > 0. \end{aligned} \tag{40}$$

Let us take just a partial sum of the infinite series (39),

$$\Phi_+^N(\lambda) = \sum_{n=0}^{N-1} \lambda^{n+1}\varphi'_n + \lambda\Gamma, \tag{41}$$

and check the validity of equation of motion (38) in a weak sense on the states φ_m[4]:

$$\left\langle\!\left\langle \varphi_m, Q\Phi_+^N(\lambda) + \Phi_+^N(\lambda) \star \Phi_+^N(\lambda)\right\rangle\!\right\rangle \tag{42}$$

$$\varphi_m = \frac{2}{\pi} c_1|0\rangle \star |m\rangle \star B_1^L c_1|0\rangle. \tag{43}$$

We use correlators [15] collected in the table below:

$$\begin{aligned} \langle\langle \varphi_m, Q\varphi_n\rangle\rangle &= -\frac{m+n+2}{\pi^2}, \\ \langle\langle \varphi_m, Q\Gamma\rangle\rangle &= \frac{1}{\pi^2}, \\ \langle\langle \Gamma, Q\Gamma\rangle\rangle &= 0, \\ \langle\langle \varphi_k, \varphi_m \star \varphi_n\rangle\rangle &= 0, \\ \langle\langle \Gamma, \varphi_m \star \varphi_n\rangle\rangle &= \frac{m+n+3}{2\pi^2}, \\ \langle\langle \Gamma, \Gamma \star \varphi_n\rangle\rangle &= 0, \\ \langle\langle \Gamma, \Gamma \star \Gamma\rangle\rangle &= 0. \end{aligned} \tag{44}$$

We get

$$\left\langle\!\left\langle \varphi_m, Q\Phi_+^N(\lambda) + \Phi_+^N(\lambda) \star \Phi_+^N(\lambda)\right\rangle\!\right\rangle = \frac{\lambda^{N+1}}{\pi^2}. \tag{45}$$

Taking the limit $N \to \infty$ for $\lambda < 1$ we have for an arbitrary m:

$$\langle\langle \varphi_m, Q\Phi_+(\lambda) + \Phi_+(\lambda) \star \Phi_+(\lambda)\rangle\rangle = 0, \tag{46}$$

in other words for $\lambda < 1$ the field $\Phi_+(\lambda)$ solves the equation of motion when contracted with states from the subspase $\mathcal{L}(\{\varphi_m\})$ spanned by φ_m. This fact is natural for the solution obtained by the iteration procedure (see section 3). It is interesting to note that if we consider the validity of the equation of motion on the subspace spanned by φ'_m we get that on this subspace the equation of motion is satisfied for any λ :

$$\langle\langle \varphi'_m, Q\Phi_+(\lambda) + \Phi_+(\lambda) \star \Phi_+(\lambda)\rangle\rangle = 0. \tag{47}$$

[4] Here $\langle\langle \ldots \rangle\rangle = \langle Y_{-2} \ldots \rangle$.

From equation (45) one sees that for $\lambda = 1$ the string field $\Phi_+ \equiv \Phi_+(1)$ does not solve equation of motion (38) in the week sense on $\mathcal{L}(\{\varphi_m\})$:

$$\langle \Phi_+(1), (Q\Phi_+(1) + \Phi_+(1) \star \Phi_+(1)\rangle \neq 0. \tag{48}$$

Let us recall that in the case of boson string to ensure the equation of motion in the sense (48) extra terms have been added to Φ_{bos}^N, and these extra terms provide the validity of the Sen conjecture [2, 7].

Following Erler [15] we can try to add to $\Phi_+^N \equiv \sum_{n=0}^{N-1} \varphi_n' + \Gamma$ two extra terms,

$$\Phi_+^N(c_1, c_2) = \Phi_+^N + c_1\varphi_N + c_2\varphi_N', \tag{49}$$

and find c_1 and c_2 from the requirement of the validity of the equation of motion in the weak sense,

$$\langle\langle \varphi_m, Q\Phi_+^N(c_1, c_2) + \Phi_+^N(c_1, c_2) \star \Phi_+^N(c_1, c_2)\rangle\rangle = 0. \tag{50}$$

Simple calculations based on (44) show that $c_1 = -1$ and c_2 is arbitrary. Indeed,

$$\begin{aligned} \langle\langle \varphi_m, Q\Phi_+^N(c_1, c_2)\rangle\rangle &= -\frac{N-1}{\pi^2} - c_1\frac{m+N+2}{\pi^2} - c_2\frac{1}{\pi^2}, \\ \langle\langle \varphi_m, \Phi_+^N(c_1, c_2) \star \Phi_+^N(c_1, c_2)\rangle\rangle &= \frac{N}{\pi^2} + c_1\frac{m+N+3}{\pi^2} + c_2\frac{1}{\pi^2}, \end{aligned} \tag{51}$$

and we see that

$$\begin{aligned} &\langle\langle \varphi_m, Q\Phi_+^N(c_1, c_2) + \Phi_+^N(c_1, c_2) \star \Phi_+^N(c_1, c_2)\rangle\rangle \\ &\quad = -\frac{N-1}{\pi^2} - c_1\frac{m+N+2}{\pi^2} - c_2\frac{1}{\pi^2} + \frac{N}{\pi^2} + c_1\frac{m+N+3}{\pi^2} + c_2\frac{1}{\pi^2} \\ &\quad = \frac{1}{\pi^2} + c_1\frac{1}{\pi^2} \end{aligned} \tag{52}$$

is equal to zero for $c_1 = -1$.

Let us add to our subspace $\mathcal{L}(\{\varphi_m\})$ a vector Γ and consider the requirement of the validity of the equations of motion also on this vector:

$$\langle\langle \Gamma, Q\Phi_+^N(-1, c_2) + \Phi_+^N(-1, c_2) \star \Phi_+^N(-1, c_2)\rangle\rangle = 0. \tag{53}$$

We have

$$\langle\langle \Gamma, Q\Phi_+^N(-1, c_2) + \Phi_+^N(-1, c_2) \star \Phi_+^N(-1, c_2)\rangle\rangle = -\frac{1}{\pi^2} + \frac{3}{2\pi^2} - c_2\frac{1}{\pi^2}, \tag{54}$$

and we see that the LHS of (54) is zero for $c_2 = 1/2$.

It is interesting to note that $c_1 = -1, c_2 = 1/2$ provide the validity of the equation of motion being contracted with $\Phi_+^N(-1, 1/2)$:

$$\langle\langle \Phi_+^N(-1, 1/2), Q\Phi_+^N(-1, 1/2) + \Phi_+^N(-1, 1/2) \star \Phi_+^N(-1, 1/2)\rangle\rangle = 0. \tag{55}$$

Therefore, we see that just the requirement of the validity of the equation of motion 'terms by terms' at the point $\lambda = 1$ forces one to add two extra terms to Φ_+^N. A necessity of these extra terms has been advocated in [15] to provide the Sen conjecture.

5. Equivalence of BSZ and ABKM theories

The action for cubic NS string theory with GSO(−) sector is presented in section 2. In the nonpolynomial theory, the GSO(−) sector can be added in the following way [11]. The field is an element of a 2 × 2 matrix of the form

$$\widehat{G} = G_+ \otimes I + G_- \otimes \sigma_1. \tag{56}$$

An equation of motion has the following form:

$$\widehat{\eta}_0(\widehat{G}^{-1}\widehat{Q}\widehat{G}) = 0, \tag{57}$$

where $\widehat{\eta}_0 \equiv \eta \otimes \sigma_3$. From a condition $\widehat{G}^{-1}\widehat{G} = I$, we have

$$G_+G_- = G_-G_+. \tag{58}$$

Let $\mathfrak{A}$ be a set of matrix solutions of equation of motion (14) and $\mathfrak{B}$ is a set of solutions (57).

Let us define a map g of $\mathfrak{B}$ to $\mathfrak{A}$ *ADD fk* [14]:

$$g : \widehat{G} \to \widehat{\Psi} \equiv g(\widehat{G}) = \widehat{G}^{-1}\widehat{Q}\widehat{G}. \tag{59}$$

This map is correctly defined due to (57). This expression in the components has the form

$$\begin{aligned} \Psi_+ &= G_+^{-1}QG_+ + \frac{G_-}{G_+^2}QG_-, \\ \Psi_- &= G_+^{-1}QG_- + \frac{G_-}{G_+^2}QG_+. \end{aligned} \tag{60}$$

In order to $\widehat{\Psi} = g(\widehat{G})$ be a solution of equation of motion (14), it is necessary and sufficient to implement the Leibnitz rule for the operator $\widehat{Q}$ (12). Let us note that G_+ is even and G_- is odd, i.e. G_+ and G_- have the different parities.

Let us define a map h of $\mathfrak{A}$ in $\mathfrak{B}$ as [14]:

$$h : \widehat{\Psi} \to \widehat{G} \equiv h(\widehat{\Psi}) = \mathrm{e}^{\widehat{P}\widehat{\Psi}}. \tag{61}$$

If $\widehat{P}^2 = 0$ we have

$$\mathrm{e}^{\widehat{P}\widehat{\Psi}} = 1 + \widehat{P}\widehat{\Psi}, \tag{62}$$

here 1 is an identity state $|I\rangle \otimes I$ with respect to $\star$, and $\widehat{P} \equiv P \otimes \sigma_3$, where P is the nilpotent operator with respect to $\star$ defined in [14],

$$(P\Psi_1) \star (P\Psi_2) = 0, \tag{63}$$

and its anticommutator with Q is the identity,

$$\{Q, P(z)\} = 1. \tag{64}$$

In the components (61) reads

$$G_+ = 1 + P\Psi_+ = \mathrm{e}^{P\Psi_+}, \qquad G_- = P\Psi_-. \tag{65}$$

$\widehat{G}^{-1}$ has the form

$$\widehat{G}^{-1} = (1 - P\Psi_+) \otimes I - P\Psi_- \otimes \sigma_1 = G_+^{-1} \otimes I - G_- \otimes \sigma_1 = \mathrm{e}^{-\widehat{P}\widehat{\Psi}}. \tag{66}$$

The maps g and h are connected nontrivially. Let us consider a composition $g \circ h$:

$$\begin{aligned} \widetilde{\widehat{\Psi}} &= (g \circ h)(\widehat{\Psi}) = g(h(\widehat{\Psi})) = (1 - \widehat{P}\widehat{\Psi})\widehat{Q}(1 + \widehat{P}\widehat{\Psi}) = (1 - \widehat{P}\widehat{\Psi})\widehat{Q}\widehat{P}\widehat{\Psi} \\ &= (1 - \widehat{P}\widehat{\Psi})(1 - \widehat{P}\widehat{Q})\widehat{\Psi} = (1 - \widehat{P}\widehat{Q} - \widehat{P}\widehat{\Psi})\widehat{\Psi} = \widehat{\Psi} - \widehat{P}(\widehat{Q}\widehat{\Psi} + \widehat{\Psi}^2) = \widehat{\Psi}, \end{aligned} \tag{67}$$

here we used (64), then we used the equation of motion for $\widehat{\Psi}$ and the nilpotency of $\widehat{P}$ under the star product (63). So we have proved that $g \circ h = Id$ and $g(\mathfrak{B}) = \mathfrak{A}$ i.e. an arbitrary classical solution in cubic theory can be represent in the pure gauge form.

Now let us consider a composition $h \circ g$:

$$\begin{aligned} \widetilde{\widehat{G}} &= (h \circ g)(\widehat{G}) = h(g(\widehat{G})) = \mathrm{e}^{\widehat{P}\widehat{G}^{-1}\widehat{Q}\widehat{G}} = 1 + \widehat{P}\widehat{G}^{-1}\widehat{Q}\widehat{G} = 1 - \widehat{P}\widehat{Q}\widehat{G}^{-1} \cdot \widehat{G} \\ &= 1 - (1 - \widehat{Q}\widehat{P})\widehat{G}^{-1} \cdot \widehat{G} = 1 - 1 + \widehat{Q}\widehat{P}\widehat{G}^{-1} \cdot \widehat{G} = \widehat{Q}\widehat{P}\widehat{G}^{-1} \cdot \widehat{G}. \end{aligned} \tag{68}$$

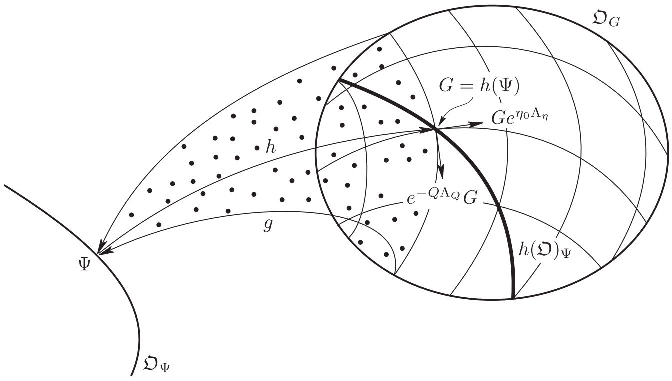

Figure 1. Maps h and g. Here hats are omitted for simplicity.

For an arbitrary $\widehat{G} \in \mathfrak{B}$ introduces the following parametrization [14]:

$$\widehat{G} = \frac{1}{1-\widehat{\Phi}}. \tag{69}$$

The element $\widehat{\Psi} = g(\widehat{G}) \in \mathfrak{A}$ takes the form

$$\widehat{\Psi} = \widehat{G}^{-1}\widehat{Q}\widehat{G} = -\widehat{Q}\widehat{G}^{-1}\widehat{G} = \widehat{Q}\widehat{\Phi}\frac{1}{1-\widehat{\Phi}}. \tag{70}$$

Here we used that $\widehat{G}$ is even, the Leibnitz rule is used, and at the same time it is important that the parities of G_+ and G_- are opposite and $\sigma_2 I = I\sigma_2, \sigma_3\sigma_1 = -\sigma_1\sigma_3$. Also we used that P changes the parity of field.

Then we use the parametrization (69) for (68):

$$\widetilde{\widehat{G}} = \widehat{Q}\widehat{P}\widehat{G}^{-1}\cdot\widehat{G} = \widehat{Q}\widehat{P}(1-\widehat{\Phi})\cdot\frac{1}{1-\widehat{\Phi}} = \frac{1}{1-\widehat{\Phi}} - \widehat{Q}\widehat{P}\widehat{\Phi}\frac{1}{1-\widehat{\Phi}} = (1-\widehat{Q}(\widehat{P}\widehat{\Phi}))\widehat{G}, \tag{71}$$

where we use

$$\widehat{Q}\widehat{P}I = I. \tag{72}$$

Let us rewrite (71) as

$$\widetilde{\widehat{G}} = \mathrm{e}^{-\widehat{Q}(\widehat{P}\widehat{\Phi})}\widehat{G}. \tag{73}$$

It is the gauge transformation

$$\widetilde{\widehat{G}} = \mathrm{e}^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}\,\mathrm{e}^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}, \tag{74}$$

with a gauge parameter $\widehat{\Lambda}_{\widehat{Q}} = \widehat{P}\widehat{\Phi}$, $\widehat{\Lambda}_{\widehat{\eta}} = 0$.

So $(h \circ g)(\widehat{G})$ belongs to a gauge orbit $\mathfrak{O}_{\widehat{G}} = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = \mathrm{e}^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}\}$ of the initial field $\widehat{G}$. In the components (73) reads

$$\begin{aligned}\widetilde{G}_+ &= \mathrm{e}^{-Q\Lambda_+}G_+ - Q\Lambda_-G_-,\\ \widetilde{G}_- &= \mathrm{e}^{-Q\Lambda_+}G_- - Q\Lambda_-G_+,\end{aligned} \tag{75}$$

$$\widehat{\Lambda}_Q = \Lambda_+ \otimes \sigma_3 + \Lambda_- \otimes \mathrm{i}\sigma_2, \tag{76}$$

where $\Lambda_+ = P\Phi_+$, $\Lambda_- = P\Phi_-$.

In terms of gauge orbits the maps g and h can be describe more clearly.

Let $\widehat{\Psi}$ be an arbitrary field of $\mathfrak{A}$ and $\widehat{G} = h(\widehat{\Psi})$. Let us consider an image of the orbit $\mathfrak{O}_{\widehat{\Psi}} = \{\widetilde{\widetilde{\widehat{\Psi}}} : \widetilde{\widetilde{\widehat{\Psi}}} = \mathrm{e}^{-\widehat{\Lambda}}(\widehat{\Psi} + \widehat{Q})\,\mathrm{e}^{\widehat{\Lambda}}\}$ by the map h: $h(\mathfrak{O}_{\widehat{\Psi}}) = \{\widetilde{\widetilde{\widehat{G}}} : \widetilde{\widetilde{\widehat{G}}} = h(\widetilde{\widetilde{\widehat{\Psi}}})\}$. The straightforward calculation gives

$$\begin{aligned}
\widetilde{\widetilde{\widehat{G}}} &= 1 + \widehat{P}\widetilde{\widetilde{\widehat{\Psi}}} = 1 + \widehat{P}(\,\mathrm{e}^{-\widehat{\Lambda}}(\widehat{\Psi} + \widehat{Q})\,\mathrm{e}^{\widehat{\Lambda}}) = 1 + \widehat{P}(-\widehat{Q}\,\mathrm{e}^{-\widehat{\Lambda}} + \mathrm{e}^{-\widehat{\Lambda}}\widehat{\Psi})\,\mathrm{e}^{\widehat{\Lambda}} \\
&= (\widehat{Q}(P\,\mathrm{e}^{-\widehat{\Lambda}}) + \widehat{P}\,\mathrm{e}^{-\widehat{\Lambda}}\widehat{\Psi})\,\mathrm{e}^{\widehat{\Lambda}} = \widehat{Q}(\widehat{P}\,\mathrm{e}^{-\widehat{\Lambda}})(1 + \widehat{P}\widehat{\Psi})\,\mathrm{e}^{\widehat{\Lambda}} = \widehat{Q}(\widehat{P}\,\mathrm{e}^{-\widehat{Q}\widehat{P}\widehat{\Lambda}})\widehat{G}\,\mathrm{e}^{\widehat{\Lambda}} \\
&= \mathrm{e}^{-\widehat{Q}\widehat{P}\widehat{\Lambda}}\widehat{G}\,\mathrm{e}^{\widehat{\Lambda}} = \mathrm{e}^{-\widehat{Q}\widehat{P}\widehat{\Lambda}}\widehat{G}\,\mathrm{e}^{\widehat{\eta}_0\widehat{\xi}\widehat{\Lambda}},
\end{aligned} \tag{77}$$

i.e. $h(\mathfrak{O}_{\widehat{\Psi}})$ is the suborbit of the field $\widehat{G} = h(\widehat{\Psi})$, due to a special choice of the gauge parameter $\widehat{\Lambda}_{\widehat{Q}}$, $\widehat{\Lambda}_{\widehat{\eta}}$ or $h(\mathfrak{O}_{\widehat{\Psi}}) \subset \mathfrak{O}_{\widehat{G}}$.

Let $\widehat{G}$ be an arbitrary field of $\mathfrak{B}$ and $\widehat{\Psi} = g(\widehat{G})$. Let us consider an image of the orbit $\mathfrak{O}_{\widehat{G}}$ by the map g: $g(\mathfrak{O}_{\widehat{G}}) = \{\widetilde{\widetilde{\widehat{\Psi}}} = g(\widetilde{\widetilde{\widehat{G}}}) : \widetilde{\widetilde{\widehat{G}}} \in \mathfrak{O}_{\widehat{G}}\}$:

$$\begin{aligned}
\widetilde{\widetilde{\widehat{\Psi}}} &= \widetilde{\widetilde{\widehat{G}}}^{-1}\widehat{Q}\widetilde{\widetilde{\widehat{G}}} = \mathrm{e}^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\widehat{G}^{-1}\,\mathrm{e}^{\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{Q}(\,\mathrm{e}^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}\,\mathrm{e}^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}) \\
&= \mathrm{e}^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\widehat{G}^{-1}((\widehat{Q}\widehat{G})\,\mathrm{e}^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}} + \widehat{G}\widehat{Q}\,\mathrm{e}^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}) = \mathrm{e}^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}(\widehat{\Psi} + \widehat{Q})\,\mathrm{e}^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}},
\end{aligned} \tag{78}$$

since $\widehat{\Lambda}_{\widehat{\eta}}$ is arbitrary, then $g(\mathfrak{O}_{\widehat{G}}) = \mathfrak{O}_{\widehat{\Psi}}$. Note that, if $h(\widehat{\Psi}') \in \mathfrak{O}_{h(\widehat{\Psi})}$, then $\widehat{\Psi}' \in \mathfrak{O}_{\widehat{\Psi}}$. Indeed, by virtue of $g \circ h = Id$ it is possible to rewrite $\widehat{\Psi}' = g(h(\widehat{\Psi}'))$, and since $g(\mathfrak{O}_{\widehat{G}}) = \mathfrak{O}_{\widehat{\Psi}}$ then $h(\widehat{\Psi}') \in \mathfrak{O}_{h(\widehat{\Psi})}$.

So we can see that the maps g and h could be constrict to the maps orbits:

$$h : \mathfrak{O}_{\widehat{\Psi}} \to \mathfrak{O}_{\widehat{G}}, \qquad g : \mathfrak{O}_{\widehat{G}} \to \mathfrak{O}_{\widehat{\Psi}}.$$

At the same time, the image $\mathfrak{O}_{\widehat{\Psi}}$ in $\mathfrak{O}_{\widehat{G}}$ is the suborbit (1). The image $\mathfrak{O}_{\widehat{G}} = \{\widetilde{\widetilde{\widehat{G}}} : \widetilde{\widetilde{\widehat{G}}} = \mathrm{e}^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}\,\mathrm{e}^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\}$ is all orbit $\mathfrak{O}_{\widehat{\Psi}}$. All elements $\mathfrak{O}_{\widehat{G}}$ with different $\widehat{\Lambda}_{\widehat{Q}}$ are mapped in one element $\mathfrak{O}_{\widehat{\Psi}}$ (see (78)). Bounded on $h(\mathfrak{O}_{\widehat{\Psi}})$ mapping g becomes invertible: $h \circ g|_{h(\mathfrak{O}_{\widehat{\Psi}})} = Id$. The composition $h \circ g$ gives in the orbit $\mathfrak{O}_{\widehat{G}}$ a special section (68).

6. Conclusion

In this paper, a singular limit of the pure gauge solution is discussed. We propose a simple prescription to deal with a singularity problem and show that it gives the same answer as the requirement of the validity of equations of motion contracted with the solutions.

The equivalence of the solutions of the equation of motion in the cubic fermionic string field theory [8] and that of nonpolynomial string field theory [11] including the GSO(-) sector is discussed using the matrix representations of both theories. The singularity problem shows once again that a formal gauge equivalence of two theories needs a rather delicate study.

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